MTH 166

Lecture-30

Line Integral

Unit 6: Vector Calculus-II

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-15**)

Topic:

Vector Integral Calculus

Learning Outcomes:

1. To calculate Line integral w.r.t to an arc length

2. To calculate Line integral of a vector field (Work Done)

Line Integral w.r.t an Arc Length

Let the parametric representation of curve C be:

$C: x = x(t), y = y(t), z = z(t); \quad a \leq t \leq b$.

Let $f(x, y, z)$ be a scalar field function.

Then line integral of scalar function f over curve C w.r.t arc length s is given by:

$$I = \int_C f(x, y, z) ds$$

$$= \int_{t=a}^{t=b} \left[f(x(t), y(t), z(t)) \frac{ds}{dt} \right] dt$$

$$= \int_{t=a}^{t=b} \left[f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \right] dt$$

Problem 1. Evaluate $\int_C f(x,y,z)ds$ where $f(x,y,z) = 2x + 3y$ and curve C is:

$C: x = t, y = 2t, z = 3t, 0 \le t \le 3.$

Solution. $C: x = t,\ y = 2t,\ z = 3t$

$$\Rightarrow \frac{dx}{dt} = 1, \quad \frac{dx}{dt} = 2\,, \quad \frac{dx}{dt} = 3$$

$$f(x,y,z) = 2x + 3y \qquad \Rightarrow f(t) = 2(t) + 3(2t) = 8t$$

Then line integral of scalar function f over curve C w.r.t arc length s is given by:

$$I = \int_C f(x,y,z)ds = \int_{t=a}^{t=b} \left[f(x(t), y(t), z(t)) \frac{ds}{dt} \right] dt$$

$$= \int_{t=a}^{t=b} \left[f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \right] dt$$

$$I = \int_{t=0}^{t=3} \left[8t\sqrt{(1)^2 + (2)^2 + (3)^2}\right] dt$$

$$= 8\sqrt{14} \int_{t=0}^{t=3} t dt$$

$$= 8\sqrt{14} \left[\frac{t^2}{2}\right]_0^3$$

$$= 8\sqrt{14} \left[\frac{9}{2} - 0\right]$$

$= 36\sqrt{14}$ **Answer.**

Problem 2. Evaluate $\int_C f(x,y)ds$ where $f(x,y,z) = x^2y$ and curve C is:

$C: x = 3\cos t\,, y = 3\sin t\,, 0 \leq t \leq \frac{\pi}{2}.$

Solution. $C: x = 3\cos t\,,\ y = 3\sin t$

$$\Rightarrow \frac{dx}{dt} = -3\sin t\,, \quad \frac{dy}{dt} = 3\cos t$$

$f(x,y) = x^2y \quad \Rightarrow f(t) = (3\cos t)^2(3\sin t) = 27cos^2 t\sin t$

Then line integral of scalar function f over curve C w.r.t arc length s is given by:

$$I = \int_C f(x,y)ds = \int_{t=a}^{t=b}\left[f(x(t),y(t))\frac{ds}{dt}\right]dt$$

$$= \int_{t=a}^{t=b}\left[f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right]dt$$

$$I = \int_{t=0}^{t=\frac{\pi}{2}} 27cos^2t\sin t\left[\sqrt{(-3\sin t)^2 + (3\cos t)^2}\right]dt$$

$$= 27 \times 3\int_{t=0}^{t=\frac{\pi}{2}} cos^2t\sin t\, dt$$

$$= -81\int_{t=0}^{t=\frac{\pi}{2}} cos^2t\,(-\sin t)\, dt$$

$$= -81\left[\frac{(\cos t)^3}{3}\right]_0^{\frac{\pi}{2}}$$

$$= -27\left[\left(\cos\frac{\pi}{2}\right)^3 - (\cos 0)^3\right]$$

$= -27(0 - 1) = 27$ **Answer.**

Line Integral of a Vector Field (Work Done)

Let us consider a vector field $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

We write $\overrightarrow{v(t)} = v_1(t)\hat{\imath} + v_2(t)\hat{\jmath} + v_3(t)\hat{k}$; $a \le t \le b$

Let curve C is represented by: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + \mathrm{z}\hat{k}$

We write C: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + \mathrm{z}(t)\hat{k}$

Then line integral of vector $\vec{v}$ over curve C given by:

$$I = \int_C \vec{v}.\overrightarrow{dr} = \int_C \left[\overrightarrow{v(t)}.\frac{\overrightarrow{dr}}{dt}\right] dt$$

$$= \int_{t=a}^{t=b} \left[\left(v_1(t)\hat{\imath} + v_2(t)\hat{\jmath} + v_3(t)\hat{k}\right).\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)\right] dt$$

$$= \int_{t=a}^{t=b} \left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dx}{dt} + v_3(t)\frac{dx}{dt}\right)\right] dt$$ This also called as **Work Done.**

Problem 1. Evaluate $\int_C \vec{v}.\overrightarrow{dr}$ where $\vec{v} = xy\hat{\imath} + y^2\hat{\jmath} + e^z\hat{k}$ and curve c is given by:

$C: x = t^2, y = 2t, z = t, 0 \leq t \leq 1.$

Solution. We write C: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $\Rightarrow \overrightarrow{r(t)} = t^2\hat{\imath} + 2t\hat{\jmath} + t\hat{k}$

$$\Rightarrow \frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \frac{dz}{dt} = 1$$

$$\vec{v} = xy\hat{\imath} + y^2\hat{\jmath} + e^z\hat{k} \qquad \Rightarrow \overrightarrow{v(t)} = 2t^3\hat{\imath} + 4t^2\hat{\jmath} + e^t\hat{k}$$

Then line integral of vector $\vec{v}$ over curve C given by:

$$I = \int_C \vec{v}.\overrightarrow{dr} = \int_C \left[\overrightarrow{v(t)}.\frac{\overrightarrow{dr}}{dt}\right] dt$$

$$= \int_{t=a}^{t=b} \left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dx}{dt} + v_3(t)\frac{dx}{dt}\right)\right] dt$$

$$I = \int_{t=a}^{t=b} \left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dx}{dt} + v_3(t)\frac{dx}{dt}\right)\right] dt$$

$$= \int_{t=0}^{t=1} \left[\left((2t^3)(2t) + (4t^2)(2) + (e^t)(1)\right)\right] dt$$

$$= \int_{t=0}^{t=1} [4t^4 + 8t^2 + e^t] dt$$

$$= \left[\frac{4}{5}t^5 + \frac{8}{3}t^3 + e^t\right]_{t=0}^{t=1}$$

$$= \left[\frac{4}{5} + \frac{8}{3} + e\right] - [0 + 0 + 1]$$

$= \frac{37}{15} + e$ **Answer.**

Problem 2. Evaluate $\int_C \vec{v}.\overrightarrow{dr}$ where $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and C is the line segment from (1,2,2) to (3,6,6). **Or.**

Find work done by force $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ in moving a particle from (1,2,2) to (3,6,6).

Solution. Two point form of line C is given by:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t \text{ (Say)}$$

$$\Rightarrow \frac{x-1}{3-1} = \frac{y-2}{6-2} = \frac{z-2}{6-2} = t$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-2}{4} = t$$

$$\Rightarrow x = 2t+1,\ y = 4t+2,\ z = 4t+2, (when\ x = 1, t = 0\ and\ x = 3, t = 1)$$

So, C: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \Rightarrow \overrightarrow{r(t)} = (2t+1)\hat{\imath} + (4t+2)\hat{\jmath} + (4t+2)\hat{k}$

$$\Rightarrow \frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 4, \quad \frac{dz}{dt} = 4$$

$$\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \Rightarrow \overrightarrow{r(t)} = (2t+1)\hat{\imath} + (4t+2)\hat{\jmath} + (4t+2)\hat{k}$$

Then line integral of vector $\vec{v}$ over curve C given by:

$$I = \int_C \vec{v}.\overrightarrow{dr} = \int_C \left[\overrightarrow{v(t)}.\frac{\overrightarrow{dr}}{dt}\right] dt$$

$$I = \int_{t=a}^{t=b} \left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dx}{dt} + v_3(t)\frac{dx}{dt}\right)\right] dt$$

$$= \int_{t=0}^{t=1} \left[\left((2t+1)(2) + (4t+2)(4) + (4t+2)(4)\right)\right] dt$$

$$= \int_{t=0}^{t=1} [36t + 18] dt = \left[36\frac{t^2}{2} + 18t\right]_{t=0}^{t=1} = (18+18) - 0 = 36$$ **Answer.**